SOME REMARKS ON MINIMAL SUFFICIENCY

by

Minoru Hasegawa and Michael D. Perlman[1]

University of Minnesota

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List of Symbols

Greek letters used: α, σ, λ, μ, τ, ρ, γ, π, Γ, Π

Script letters used: $\mathcal{B}$, $\mathcal{L}$, $\mathcal{h}$, $\mathcal{P}$, $\mathcal{S}$, $\mathcal{X}$

Special symbols used: $\forall$, $\int$, $\ll$, $\subseteq$, $\cup$, $\cap$, $\vee$, $\leq$, ∞, $\in$ (= "element of"),

$\wedge$, $<$, $\Leftrightarrow$, $=$, $\equiv$

ABSTRACT

Some Remarks on Minimal Sufficiency

Let $\mathcal{P} = \{P\}$ be a family of probability measures such that either each P is dominated by a fixed σ-finite measure μ or each P is a discrete measure. It is shown directly that the intersection of an arbitrary (not necessarily countable) collection of sufficient subfields is sufficient, provided that each subfield contains all $\mathcal{P}$-null sets. This provides an alternate demonstration of the existence of a minimal sufficient subfield. Also, for a general family $\mathcal{P}$, it is proved that if a subfield is sufficient and boundedly complete for $\mathcal{P}$, then it is minimal sufficient.

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1. Minimal Sufficiency in the Dominated Case.

Let $(\mathfrak{X}, \mathcal{S})$ be a measurable space (the sample space) and $\mathcal{P} = \{P\}$ a family of probability measures on $\mathcal{S}$. Suppose that each P is dominated by a fixed σ-finite measure μ. Let $\mathfrak{h}$ be the sub-σ-field (subfield) of $\mathcal{S}$ consisting of all $\mathcal{P}$-null sets and their complements. For two subfields $\mathcal{S}_0$, $\mathcal{S}_1$ write $\mathcal{S}_0 \subseteq \mathcal{S}_1[\mathcal{P}]$ if for each $S_0 \in \mathcal{S}_0$, there exists $S_1 \in \mathcal{S}_1$ such that the symmetric difference $(S_0 - S_1) \cup (S_1 - S_0)$ is $\mathcal{P}$-null. Then $\mathcal{S}_0 \subseteq \mathcal{S}_1[\mathcal{P}] \Leftrightarrow \mathcal{S}_0 \subseteq \mathcal{S}_1 \vee \mathfrak{h}$ (the smallest subfield containing $\mathcal{S}_0$ and $\mathfrak{h}$. Following Bahadur (1954), we say a subfield $\mathcal{S}_0$ is <u>minimal</u> <u>sufficient</u> for $\mathcal{P}$ if $\mathcal{S}_0$ is sufficient for $\mathcal{P}$ and if $\mathcal{S}_0 \subseteq \mathcal{S}_1[\mathcal{P}]$ for any other sufficient subfield $\mathcal{S}_1$.

In the dominated case, a minimal sufficient subfield always exists-- one such subfield $\mathcal{S}^*$ is constructed by Bahadur (1954, Theorem 6.2). From this and Theorem 6.4 of Bahadur (1954) it follows that the intersection of an arbitrary collection $\{\mathcal{S}_\alpha\}$ of sufficient subfields is sufficient, provided that each $\mathcal{S}_\alpha$ contains $\mathfrak{h}$: $\mathcal{S}^* \subseteq \mathcal{S}_\alpha \vee \mathfrak{h} = \mathcal{S}_\alpha$ for each α, so $\mathcal{S}^* \subseteq \cap \mathcal{S}_\alpha$, hence $\cap \mathcal{S}_\alpha$ is sufficient.

In this section we point out the fact, of some pedagogical interest, that the order of these two steps can be reversed. First we give a direct proof of the fact that the intersection of sufficient subfields containing $\mathfrak{h}$ is sufficient, based on a martingale convergence theorem of Krickeberg. From this the existence of a minimal sufficient subfield is easily deduced.

<u>Theorem 1.1.</u>

Let $\{\mathcal{S}_\alpha\}$ be an arbitrary collection of sufficient subfields with $\mathfrak{h} \subseteq \mathcal{S}_\alpha$ for each α. Then $\mathcal{S}_{-\infty} \equiv \cap \mathcal{S}_\alpha$ is also sufficient.

To deduce the existence of a minimal sufficient subfield from Theorem 1.1, let $\Gamma = \{\mathcal{S}_\alpha\}$ be the collection of all sufficient subfields which contain $\mathfrak{h}$. This collection is non-empty since it contains $\mathcal{S}$, so $\mathcal{S}_{-\infty} = \cap \mathcal{S}_\alpha$ is sufficient. If $\mathcal{S}_1$ is any other sufficient subfield, it is easy to see that $\mathcal{S}_1 \vee \mathfrak{h}$ is also sufficient and belongs to Γ, so $\mathcal{S}_{-\infty} \subseteq \mathcal{S}_1 \vee \mathfrak{h}$, i.e., $\mathcal{S}_{-\infty} \subseteq \mathcal{S}_1[\mathcal{P}]$, hence $\mathcal{S}_{-\infty}$ is minimal sufficient.

The proof of Theorem 1.1 is based on the following three lemmas, which are (respectively) a well-known factorization criterion for sufficiency in the dominated case, a version of Theorem 1.1 for a finite collection $\{\mathcal{S}_\alpha\}$, and a generalized reverse martingale convergence theorem of Krickeberg (1960). Let $\{\mathcal{P}_i\}_{i=1}^\infty$ be a countable subfamily of $\mathcal{P}$ which is equivalent to $\mathcal{P}$ (Halmos and Savage (1949), Lemma 7) and let $\lambda = \Sigma 2^{-i} P_i$.

Lemma 1.1 (Halmos-Savage, Bahadur).

A subfield $\mathcal{S}_0$ is sufficient for $\mathcal{P}$ $\Leftrightarrow$ for each $P \in \mathcal{P}$ there exists an $\mathcal{S}_0$-measurable version of the Radon-Nikodym derivative $dP/d\lambda$.

Lemma 1.2. (Burkholder).

Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be two sufficient subfields such that at least one contains $\mathfrak{h}$. Then $\mathcal{S}_0 \cap \mathcal{S}_1$ is also sufficient for $\mathcal{P}$.

Proof:

This result is true even without the assumption that $\mathcal{P}$ is dominated (Burkholder (1961), Theorem 4). Under this assumption, however, a simple proof is possible, which we now present. Suppose $\mathfrak{h} \subseteq \mathcal{S}_1$. For a fixed $P \in \mathcal{P}$, let f_i be an $\mathcal{S}_i$-measurable version of $dP/d\lambda$, i = 1, 2 (by Lemma 1.1). Then

$$\int_S f_0 d\lambda = P(S) = \int_S f_1 d\lambda$$

for all $S \in \mathcal{S}$, so $f_0 = f_1$ a.e. $[\lambda]$. Since λ is equivalent to $\mathcal{P}$, this implies that $f_0 - f_1$ is $\mathfrak{h}$-measurable, hence $\mathcal{S}_1$-measurable. Thus $f_0 = (f_0 - f_1) + f_1$ is both $\mathcal{S}_0$ and $\mathcal{S}_1$-measurable, hence f_0 is an $(\mathcal{S}_0 \cap \mathcal{S}_1)$-measurable version of $dP/d\lambda$. Thus by Lemma 1.1, $\mathcal{S}_0 \cap \mathcal{S}_1$ is sufficient for $\mathcal{P}$.

We need the following terminology for Lemma 1.3. Let $(E, \mathcal{B}, \mu)$ be a probability space. Let T be an arbitrary index set and let $\{\mathcal{B}_\tau | \tau \in T\}$ be a non-empty collection of subfields of $\mathcal{B}$ directed downward by inclusion, i.e., for each pair $\sigma, \tau \in T$, there exists $\rho \in T$ such that $\mathcal{B}_\rho \subseteq \mathcal{B}_\sigma \cap \mathcal{B}_\tau$. Then T becomes a directed set under the partial ordering $\ll$ defined by $\rho \ll \tau \Leftrightarrow \mathcal{B}_\tau \subseteq \mathcal{B}_\rho$. Let $\mathcal{B}_{-\infty} = \cap \mathcal{B}_\tau$. For each $\tau \in T$, let f_τ be a real-valued, $\mathcal{B}_\tau$-measurable, μ-integrable function. The collection $\{(f_\tau, \mathcal{B}_\tau) | \tau \in T\}$ is said to be a (reverse) martingale relative to μ if

$$f_\tau = E_\mu[f_\rho | \mathcal{B}_\tau] \text{ a.e. } [\mu] \tag{1.1}$$

whenever $\rho \ll \tau$. The collection $\{f_\tau\}$ is said to be μ-uniformly integrable if

$$\int_{\{|f_\tau| > \gamma\}} |f_\tau| d\mu \to 0 \quad \text{as} \quad \gamma \to \infty \tag{1.2}$$

uniformly in $\tau \in T$. Let $\|f\|$ denote the $\mathcal{L}_1(\mu)$-norm of f. As a special case of Theorem 2.2 of Krickeberg (1960) we state the third lemma.

Lemma 1.3. (Krickeberg).

Let $\{(f_\tau, \mathcal{B}_\tau) | \tau \in T\}$ be a μ-uniformly integrable martingale. Then there exists a $\mathcal{B}_{-\infty}$-measurable, μ-integrable function $f_{-\infty}$ such that

$$\lim_\tau \|f_\tau - f_{-\infty}\| = 0. \tag{1.3}$$

Proof of Theorem 1.1.

Let $(E, \mathcal{B}, \mu) = (\mathfrak{X}, \mathcal{S}, \lambda)$ and let $\{\mathcal{B}_\tau\}$ be the class of all finite intersections of the given collection $\{\mathcal{S}_\alpha\}$. Then $\{\mathcal{B}_\tau\}$ is directed downward, in fact is closed under finite intersections, and $\mathcal{B}_{-\infty} = \mathcal{S}_{-\infty}$. By Lemma 1.2 each $\mathcal{B}_\tau$ is sufficient for $\mathcal{P}$, so for a fixed $P \in \mathcal{P}$ there exists a nonnegative $\mathcal{B}_\tau$-measurable version f_τ of $dP/d\lambda$ on $\mathcal{S}$. For any ρ, τ we have

$$(1.4) \qquad \int_S f_\tau d\lambda = P(S) = \int_S f_\rho d\lambda \qquad \forall S \in \mathcal{S},$$

so (1.1) is satisfied and $\{(f_\tau, \mathcal{B}_\tau)\}$ is a martingale relative to λ. Furthermore, (1.4) implies that

$$(1.5) \qquad f_\tau = f_\rho \quad \text{a.e. } [\lambda].$$

Since $\int f_\tau d\lambda = 1$ we have that

$$\int_{\{f_\tau > \gamma\}} f_\tau d\lambda \to 0 \quad \text{as} \quad \gamma \to \infty,$$

and by (1.5) this convergence is uniform in τ, so (1.2) holds and $\{f_\tau\}$ is λ-uniformly integrable. Thus $\{(f_\tau, \mathcal{B}_t)\}$ satisfies the conditions of Lemma 1.3, so there exists a $\mathcal{B}_{-\infty}(= \mathcal{S}_{-\infty})$-measurable function $f_{-\infty}$ such that (1.3) is satisfied. In particular, for each $S \in \mathcal{S}$,

$$\int_S f_{-\infty} d\lambda = \lim_\tau \int_S f_\tau d\lambda = P(S),$$

so $f_{-\infty}$ is an $\mathcal{S}_{-\infty}$-measurable version of $dP/d\lambda$. Thus by Lemma 1.1, $\mathcal{S}_{-\infty}$ is sufficient for $\mathcal{P}$, as stated.

If we drop the assumption that $\mathcal{P}$ is dominated, then Theorem 1.1 is valid if $\{\mathcal{S}_\alpha\}$ is a countable collection (Burkholder (1961), Corollary 2, p. 1197) but not (in general) if $\{\mathcal{S}_\alpha\}$ is an arbitrary collection. Otherwise, a minimal sufficient statistic would always exists, but Pitcher (1957) has shown that this is not true.

The assumption concerning $\mathfrak{N}$ cannot be dropped in Theorem 1.1 and Lemma 1.2. Burkholder (1961, Example 3) provides an example of two sufficient subfields, neither containing $\mathfrak{N}$, whose intersection is not sufficient.

2. Minimal Sufficiency in the Discrete Case.

In this section we consider minimal sufficiency in the discrete case treated by Basu and Ghosh (1969) and Morimoto (1972), and show that Theorem 1.1 continues to hold. Let $\mathcal{P} = \{P\}$ be a collection of discrete probability measures defined on a measurable space $(\mathfrak{X}, \mathcal{S})$, where now $\mathcal{S} = 2^{\mathfrak{X}}$ is the collection of all subsets of $\mathfrak{X}$. We assume that for each $x \in \mathfrak{X}$, there exists at least one $P \in \mathcal{P}$ such that $P(x) > 0$, so that $\mathfrak{N} = \{\emptyset, \mathfrak{X}\}$ only. Sufficiency and minimal sufficiency for subfields are defined exactly as before, but now $\mathcal{S}_0 \subseteq \mathcal{S}_1[\mathcal{P}] \Leftrightarrow \mathcal{S}_0 \subseteq \mathcal{S}_1$. Also, if either $\mathfrak{X}$ or $\mathcal{P}$ is countable then $\mathcal{P}$ is dominated and the methods and results of the preceding section apply.

We use the terminology of Basu and Ghosh (1969) and Morimoto (1972). A partition $\Pi = \{\pi\}$ of the sample space is a disjoint collection of subsets π whose union is $\mathfrak{X}$. Each partition Π induces a subfield $\mathcal{S}(\Pi)$ defined to be the collection of all unions (not necessarily countable) of members of Π. If a subfield $\mathcal{S}_0$ is equal to $\mathcal{S}(\Pi)$ for some partition Π, $\mathcal{S}_0$ is said to be inducible. A partition Π is sufficient (or minimal sufficient) if $\mathcal{S}(\Pi)$ has this property.

First we give some elementary definitions and facts concerning partitions. For any partition $\Pi = \{\pi\}$ and any $x \in \mathfrak{X}$, let $\pi(x)$ denote the unique member of Π which contains x. A partition Π determines an equivalence relation (Π) defined by $x(\Pi)y$ iff $y \in \pi(x)$ iff $x \in \pi(y)$. We say

Π_1 is <u>coarser</u> than Π_2 (equivalently, Π_2 is <u>finer</u> than Π_1) if each member of Π_1 is a union of members of Π_2, and write $\Pi_1 < \Pi_2$ in this case. Note that $\Pi_1 < \Pi_2$ iff $x(\Pi_2)y$ implies $x(\Pi_1)y$ for all x, y. Also,

(2.1) $\quad \Pi_1 < \Pi_2$ iff $\mathcal{S}(\Pi_1) \subseteq \mathcal{S}(\Pi_2)$.

Let $\{\mathcal{S}_\alpha\}$ be an arbitrary collection of inducible subfields, i.e., $\mathcal{S}_\alpha = \mathcal{S}(\Pi_\alpha)$ for some Π_α. Morimoto (1972) has shown that a subfield is inducible iff it is closed under arbitrary unions, so $\mathcal{S}_{-\infty} \equiv \cap \mathcal{S}_\alpha$ is also inducible, say $\mathcal{S}_{-\infty} = \mathcal{S}(\Pi_{-\infty})$ for some (unique) partition $\Pi_{-\infty}$. From (2.1), $\Pi_{-\infty} < \Pi_\alpha$ for all α. Furthermore, if $\Pi^* < \Pi_\alpha$ for all α then $\mathcal{S}(\Pi^*) \subseteq \mathcal{S}_\alpha$ for all α, hence $\mathcal{S}(\Pi^*) \subseteq \mathcal{S}_{-\infty} = \mathcal{S}(\Pi_{-\infty})$, and so $\Pi^* < \Pi_{-\infty}$. Thus $\Pi_{-\infty}$ is the (unique) finest partition coarser than every Π_α. We denote $\Pi_{-\infty}$ by $\wedge\Pi_\alpha$, so that we may write $\cap\, \mathcal{S}(\Pi_\alpha) = \mathcal{S}(\wedge\Pi_\alpha)$. We call $\wedge\Pi_\alpha$ the <u>intersection</u> of the partitions Π_α (there does not seem to be a standard terminology for $\wedge\Pi_\alpha$). The next two lemmas characterize $\wedge\Pi_\alpha$ in two important cases. The proofs are straightforward.

<u>Lemma 2.1</u>.

$x(\Pi_1 \wedge \Pi_2)y$ iff there exists a finite subset $\{z_1, \ldots, z_n\}$ of $\mathcal{X}$ such that

$$x(\Pi_1)z_1(\Pi_2)z_2(\Pi_1)z_3(\Pi_2)z_4 \ldots z_n(\Pi_i)y$$

where $i = 1$ or 2 ($n = 0$ is permitted).

<u>Lemma 2.2</u>.

Let $\{\Pi_\tau\}$ be a collection of partitions which is closed under finite intersections. Then $x(\wedge\Pi_\tau)y$ iff $x(\Pi_\tau)y$ for some τ.

In the discrete case, Basu and Ghosh (1969) have given a constructive proof of the existence of the minimal sufficient partition (necessarily unique in this case). From this and Remark 1 of Basu and Ghosh it follows that the intersection of an arbitrary collection of sufficient partitions is sufficient. In this section we show that, again, the order of these two steps may be reversed.

The main result of this section (Theorem 2.1) states that Theorem 1.1 holds in the discrete case. Because of the following lemma due to Basu and Ghosh (1969), this result can be stated in terms of partitions rather than subfields.

Lemma 2.3 (Basu and Ghosh).

A sufficient subfield is inducible.

Theorem 2.1.

Let $\{\Pi_\alpha\}$ be an arbitrary collection of sufficient partitions. Then $\Pi_{-\infty} \equiv \wedge\Pi_\alpha$ is also sufficient.

The existence of a minimal sufficient partition in the discrete case now follows by the argument given after Theorem 1.1.

To prove Theorem 2.1 two further lemmas are needed, the analogs of Lemmas 1.1 and 1.2. The first is a factorization criterion for sufficiency in the discrete case given by Basu and Ghosh (1969) (also see Ferguson (1967), p. 115).

Lemma 2.4 (Basu and Ghosh).

A partition Π is sufficient for $\mathcal{P}$ iff there exists a function $g(x)$ such that

$$P(x) = g(x)P(\pi(x)) \tag{2.2}$$

for every $P \in \mathcal{P}$ and every $x \in \mathfrak{X}$.

Remark 1.

By assumption, for each x there is some $P_x \in \mathcal{P}$ such that $P_x(x) > 0$, so $g(x) = P_x[x|\pi(x)] > 0$ for each x. Furthermore, $P_x(y) = g(y)P_x(\pi(x)) > 0$ for all $y \in \pi(x)$, so $\pi(x)$ is countable. Thus if Π is a sufficient partition, each member $\pi \in \Pi$ is countable.

Lemma 2.5 (Burkholder).

If Π_1 and Π_2 are sufficient partitions for $\mathcal{P}$ then $\Pi_1 \wedge \Pi_2$ is also sufficient for $\mathcal{P}$.

Proof:

As was the case with Lemma 1.1, this result is a consequence of Theorem 4 of Burkholder (1961). We present here a proof for the discrete case which illuminates the "alternating operator" theorem of Burkholder and Chow (1961) used by Burkholder to prove the general case. By Lemma 2.4, there exists a function $g_i(x) > 0$ such that

(2.3) $$P(x) = g_i(x)P(\pi_i(x))$$

for all x and P, where $i = 1, 2$. Let $\Pi = \Pi_1 \wedge \Pi_2$; we shall construct a function $g(x)$ satisfying (2.2). First, since each member of Π_1 and Π_2 is countable (Remark 1), it follows from Lemma 2.1 that each member of Π is countable. Fix $x \in \mathfrak{X}$ and let $\pi(x) = \{y_k\}$. By Lemma 2.1, for each $y_k \in \pi(x)$ there exists a finite subset $\{z_{kj} | j = 1,\dots, n(k)\}$ of $\mathfrak{X}$ such that

(2.4) $$y_k(\Pi_1)z_{k1}(\Pi_2)z_{k2}(\Pi_1)z_{k3} \dots z_{kn(k)}(\Pi_{i(k)})x.$$

Then by applying (2.3) and (2.4) repeatedly we find that for any P,

$$\begin{aligned} P(y_k) &= \frac{g_1(y_k)}{g_1(z_{k1})}\frac{g_2(z_{k1})}{g_2(z_{k2})}\frac{g_1(z_{k2})}{g_1(z_{k3})} \cdots \frac{g_{i(k)}(z_{kn(k)})}{g_{i(k)}(x)} P(x) \\ &\equiv h_k(x)P(x), \end{aligned}$$

say, where $0 < h_k(x) < \infty$ is independent of P. Therefore for all P,

$$P(\pi(x)) = \sum_k P(y_k) = (\sum_k h_k(x))P(x) \equiv h(x)P(x),$$

say, where $0 < h(x) < \infty$ is also independent of P. Thus (2.2) is satisfied with $g(x) = 1/h(x)$.

Proof of Theorem 2.1.

Let $\{\Pi_\tau | \tau \in T\}$ be the collection of all finite intersections of the partitions Π_α. The index set T becomes a directed set under the partial ordering $\ll$ defined by $\rho \ll \tau \Leftrightarrow \Pi_\tau < \Pi_\rho$. Clearly $\wedge\Pi_\alpha = \wedge\Pi_\tau \equiv \Pi_{-\infty}$, say, and each Π_τ is sufficient by Lemma 2.5. Let $g_\tau(x) > 0$ be chosen so that

$$P(x) = g_\tau(x)P(\pi_\tau(x)) \tag{2.5}$$

for all x and P. If $\rho \ll \tau$ then $\pi_\rho(x) \subseteq \pi_\tau(x)$, so for each fixed x and P the net $\{P(\pi_\tau(x)) | \tau \in T\}$ is monotonically increasing. Hence by (2.5), the net $\{g_\tau(x) | \tau \in T\}$ is monotonically decreasing. Therefore for all x, P,

$$\lim_\tau P(\pi_\tau(x)) = \sup_\tau P(\pi_\tau(x)) \equiv L(P, x) \tag{2.6}$$

$$\lim_\tau g_\tau(x) = \inf_\tau g_\tau(x) \equiv g(x)$$

where $g(x)$ does not depend on P. From (2.5) and (2.6)

$$P(x) = g(x)L(P, x)$$

for all x and P. Therefore by Lemma 2.4, to show that $\Pi_{-\infty}$ is sufficient it is enough to show that

$$L(P, x) = P(\pi_{-\infty}(x)). \tag{2.7}$$

Fix x and P. First, for any $\tau \in T$

$$P(\pi_{-\infty}(x)) \equiv \sum_{y \in \pi_{-\infty}(x)} P(y) \geq \sum_{y \in \pi_\tau(x)} P(y) \equiv P(\pi_\tau(x))$$

by Lemma 2.2, so

$$P(\pi_{-\infty}(x)) \geq \sup_\tau P(\pi_\tau(x)) = L(P, x).$$

Next, $\pi_{-\infty}(x)$ contains at most countably many points y such that $P(y) > 0$, say $\{y_1, y_2, \ldots\}$, so $P(\pi_{-\infty}(x)) = \Sigma P(y_k)$. Since $y_k(\Pi_{-\infty})x$, Lemma 2.2 implies that there exists $\tau_k \in T$ such that $y_k(\Pi_{\tau_k})x$, and the $\{\tau_k\}$ can be chosen so that $\tau_k \ll \tau_{k+1}$ for all k. Therefore

$$P(\pi_{-\infty}(x)) = \lim_n \sum_{k=1} P(y_k) \leq \lim_n P(\pi_{\tau_k}(x)) \leq L(P,x).$$

Hence (2.7) is verified and the proof is complete.

3. Completeness and Minimal Sufficiency.

Let $(\mathcal{X}, \mathcal{S})$ and $\mathcal{P}$ be a measurable space and a collection of probability measures on $\mathcal{S}$, respectively, with no additional restrictions. A subfield $\mathcal{S}_0$ is boundedly complete for $\mathcal{P}$ if for any bounded $\mathcal{S}_0$-measurable function f, $\int f dP = 0 \ \forall P \in \mathcal{P}$ implies that $f = 0$ a.e. $[\mathcal{P}]$. In the original study of completeness and sufficiency, Lehmann and Scheffe (1950, Theorem 3.1) show that if $\mathcal{S}_0$ is sufficient and boundedly complete for $\mathcal{P}$, and if a minimal sufficient subfield $\mathcal{S}^*$ exists, then $\mathcal{S}_0$ and $\mathcal{S}^*$ are equivalent in the sense that $\mathcal{S}_0 \subseteq \mathcal{S}^*[\mathcal{P}]$ and $\mathcal{S}^* \subseteq \mathcal{S}_0[\mathcal{P}]$, so that $\mathcal{S}_0$ is also minimal sufficient. (Actually, Lehmann and Scheffe state their result for subfields induced by statistics, but the proof essentially is the same in both cases.) Subsequently, this result often has been stated without mention of $\mathcal{S}^*$, in the following stronger form: if $\mathcal{S}_0$ is sufficient and boundedly complete for $\mathcal{P}$, then $\mathcal{S}_0$ is minimal sufficient. To our knowledge no proof of the stronger statement appears in the literature.

(This result is stated by Zacks (1971, Lemma 2.6.3) but his proof only treats the case where $\mathcal{S}_0$ is induced by a real-valued statistic, and even for this case the accuracy of the proof is questionable.) Therefore, we now present a proof of the stronger version.

<u>Theorem 3.1.</u>

If $\mathcal{S}_0$ is sufficient and boundedly complete for $\mathcal{P}$, then $\mathcal{S}_0$ is minimal sufficient for $\mathcal{P}$.

<u>Proof:</u>

Let $\mathcal{S}_1$ be another sufficient subfield; we must show $\mathcal{S}_0 \subseteq \mathcal{S}_1[\mathcal{P}]$. Fix $S_0 \in \mathcal{S}_0$. The sufficiency of $\mathcal{S}_1$ implies that there exists an $\mathcal{S}_1$-measurable function f_1 such that

(3.1) $$f_1 = E_P[I_{S_0}|\mathcal{S}_1] \quad \forall P \in \mathcal{P},$$

where I denotes the indicator function. Since $0 \leq f_1 \leq 1$ a.e. $[\mathcal{P}]$, we can modify f_1 on an $\mathcal{S}_1$-measurable $\mathcal{P}$-null set to insure that $0 \leq f_1 \leq 1$ everywhere, without affecting (3.1). Next, the sufficiency of $\mathcal{S}_0$ implies that there exists an $\mathcal{S}_0$-measurable function f_0 such that

(3.2) $$f_0 = E_P[f_1|\mathcal{S}_0] \quad \forall P \in \mathcal{P},$$

and again we can take $0 \leq f_0 \leq 1$ everywhere. Now by (3.1) and (3.2),

$$\int (f_0 - I_{S_0})dP = \int (f_1 - I_{S_0})dP = 0 \quad \forall P \in \mathcal{P},$$

and $f_0 - I_{S_0}$ is $\mathcal{S}_0$-measurable and bounded, so

(3.3) $$f_0 = I_{S_0} \text{ a.e. } [\mathcal{P}].$$

Then by (3.2) and (3.3)

$$\int I_{S_0} f_1 dP = \int I_{S_0} f_0 dP = \int I_{S_0} dP$$

$\forall P \in \mathcal{P}$. Since $0 \leq f_1 \leq 1$, this implies that

$$I_{S_0} f_1 = I_{S_0} \text{ a.e. } [\mathcal{P}]. \tag{3.4}$$

Again by (3.2) and (3.3)

$$\int I_{S_0'} f_1 dP = \int I_{S_0'} f_1 dP = 0$$

$\forall P \in \mathcal{P}$, where S_0' is the complement of S_0, which implies that

$$I_{S_0'} f_1 = 0 \text{ a.e. } [\mathcal{P}]. \tag{3.5}$$

Therefore, if we define the $\mathcal{S}_1$-measurable set

$$S_1 \equiv \{x | f_1(x) = 1\},$$

it follows from (3.4) and (3.5) that the symmetric difference of S_0 and S_1 is $\mathcal{P}$-null. Hence $\mathcal{S}_0 \subseteq \mathcal{S}_1[\mathcal{P}]$.

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